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Quantization dimension function and Gibbs measure associated with Moran set

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In this paper we consider the Gibbs measure on the one-sided shift dynamical system and determine the quantization dimension function for the image measure supported on a Moran set. A relationship between the quantization dimension function and the temperature function of the thermodynamic formalism arising in multifractal analysis is also established.

1. Introduction

The problem of quantization originally arose in electrical engineering in the process of signal processing and data compression (cf. [4]). Following the work of Graf and Luschgy, we define the *quantization dimension* (or perhaps better, the *quantization dimension function*) as follows (cf. [5,6]). Given a Borel probability measure μ on $\mathbb{R}^d$, a number $r \in (0, +\infty)$ and a natural number $n \in \mathbb{N}$, the nth *quantization error* of order r for μ, is defined by

$$e_{n,r} = \inf\left\{\left(\int d(x,\alpha)^r \, d\mu(x)\right)^{\frac{1}{r}} : \alpha \subset \mathbb{R}^d, \ \operatorname{card}(\alpha) \leqslant n\right\},$$

where $d(x,\alpha)$ denotes the distance from the point x to the set α with respect to a given norm $\|\cdot\|$ on $\mathbb{R}^d$. We note that if $\int \|x\|^r \, d\mu(x) < \infty$ then there is some set α for which the infimum is achieved (cf. [5]). The upper and lower quantization dimensions for μ of order r are defined by

$$\overline{D}_r(\mu) := \limsup_{n\to\infty} \frac{\log n}{-\log e_{n,r}}; \qquad \underline{D}_r(\mu) := \liminf_{n\to\infty} \frac{\log n}{-\log e_{n,r}}.$$

If $\overline{D}_r(\mu)$ and $\underline{D}_r(\mu)$ coincide, we call the common value the quantization dimension of μ of order r and is denoted by $D_r(\mu)$. Graf and Luschgy also defined $e_{n,r}$ where $r = 0$ and $r = +\infty$, but in the paper we only deal with the case $0 < r < +\infty$. One sees that the quantization dimension is actually a function $r \mapsto D_r$ which measures the asymptotic rate at which $e_{n,r}$ goes to zero. If D_r exists, then one can write

$$\log e_{n,r} \sim \log\left(\frac{1}{n}\right)^{1/D_r}.$$

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Let $S_1, S_2, \ldots, S_N$ be a set of contractive similarity mappings on $\mathbb{R}^d$ with the similarity ratios $s_1, s_2, \ldots, s_N$ for $N \geqslant 2$. Then for a given probability vector $(p_1, p_2, \ldots, p_N)$ there exists a unique probability measure μ (cf. [8]) supported on the self-similar set generated by the similarity mappings $S_1, \ldots, S_N$, and satisfies

$$\mu = \sum_{i=1}^{N} p_i \mu \circ S_i^{-1}. \tag{1}$$

Let the iterated function system $\{S_1, S_2, \ldots, S_N\}$ satisfy the open set condition: there exists a bounded nonempty open set $U \subset \mathbb{R}^d$ such that $\bigcup_{i=1}^N S_i(U) \subset U$ and $S_i(U) \cap S_j(U) = \emptyset$ for $1 \leqslant i \neq j \leqslant N$. Then, Graf and Luschgy showed that quantization dimension function $D_r := D_r(\mu)$ for the probability measure μ exists, and satisfies the following relation (cf. [5,7]):

$$\sum_{i=1}^{N} \left(p_i s_i^r\right)^{\frac{D_r}{r+D_r}} = 1.$$

Note that from the above relation it is clear that quantization dimension function for a self-similar probability measure has a relationship with the temperature function of the thermodynamic formalism arising in multifractal analysis (cf. [2]). For some other results relating to quantization dimension function and the temperature function, one could see [1,11,15,16]. In this paper, we consider the Gibbs measure on the one-sided shift dynamical system and determine the quantization dimension function for the image measure supported on a Moran set. Note that a self-similar set is a special case of a Moran set. In Theorem 3.1, we have also established a relationship between the quantization dimension function D_r and the temperature function $\beta(q)$ of the thermodynamic formalism arising in multifractal analysis.

2. Basic definitions, lemmas and propositions

Let us write

$$V_{n,r} = \inf\left\{ \int d(x, \alpha)^r \, d\mu(x) \colon \alpha \subset \mathbb{R}^d,\ \operatorname{card}(\alpha) \leqslant n \right\},$$
$$u_{n,r} = \inf\left\{ \int d\left(x, \alpha \cup U^c\right)^r \, d\mu(x) \colon \alpha \subset \mathbb{R}^d,\ \operatorname{card}(\alpha) \leqslant n \right\},$$

where U is a set which comes from the strong open set condition (definition follows) and U^c denotes the complement of U. We see that

$$u_{n,r}^{1/r} \leqslant V_{n,r}^{1/r} = e_{n,r}.$$

We will call sets $\alpha_n \subset \mathbb{R}^d$, for which the above infimums are achieved, n-optimal sets for $e_{n,r}$, $V_{n,r}$ or $u_{n,r}$ respectively. As stated above, Graf and Luschgy have shown that n-optimal sets exist when $\int \|x\|^r \, d\mu(x) < \infty$.

Let $\mathcal{A} = \{1, 2, \ldots, N\}$ be a finite set for $N \geqslant 2$, and M is a $\{0, 1\}$ $N \times N$ matrix, and assume that every column and row of M has a nonzero entry. Define Ω and $\sigma : \Omega \to \Omega$ by

$$\Omega = \left\{ x = (x_i)_{i=1}^{\infty} \colon M(x_i, x_{i+1}) = 1 \text{ for } i = 1, 2, \ldots \right\}$$

and

$$\sigma : \Omega \ni (x_1, x_2, x_3, \ldots) \mapsto (x_2, x_3, \ldots) \in \Omega.$$

Let $d : \Omega \times \Omega \to \mathbb{R}$ be defined by

$$d(x, y) = \sum_{i=1}^{\infty} \left(1 - \delta(x_i, y_i)\right) 2^{-(i-1)}$$

where $\delta(x_i, y_i) = 1$ if $x_i = y_i$, and $= 0$ if $x_i \neq y_i$ for $x = (x_1, x_2, \ldots),\ y = (y_1, y_2, \ldots) \in \Omega$. Then d is a metric on Ω. With this metric, Ω becomes a compact metric space and σ a positively expansive surjective local homeomorphism. The dynamical system (Ω, σ) is called the one-sided subshift of finite type (or one-sided topological Markov shift) defined by M, and σ is called the shift map. If the matrix M is primitive, i.e., there is a positive integer n such that every entry of M^n is positive, then the one-sided subshift (Ω, σ) is said to be topologically mixing.

Let (Ω, σ) be a one-sided subshift of finite type and $\mathcal{C}(\Omega)$ be the Banach space of all real valued continuous functions on Ω with the supremum norm $\|\cdot\|$. For $\phi \in \mathcal{C}(\Omega)$ and $k = 1, 2, \ldots,$ define the kth variation of ϕ by

$$\operatorname{Var}_k \phi = \sup\left\{ \left|\phi(x) - \phi(y)\right| \colon x, y \in \Omega,\ x_1 x_2 \cdots x_k = y_1 y_2 \cdots y_k \right\}.$$

Since ϕ is uniformly continuous, it follows that $\mathrm{Var}_k\,\phi \to 0$ as $k \to \infty$. The function ϕ is said to have *summable variation* if

$$\sum_{k=1}^{\infty} \mathrm{Var}_k\,\phi < \infty.$$

Let $\mathcal{S}(\Omega)$ denote the set of all functions on Ω with summable variation. If there exist $a > 0$ and $b \in (0,1)$ such that

$$\mathrm{Var}_k\,\phi \leqslant ab^{k-1} \quad (k = 1, 2, 3, \ldots)$$

then ϕ is said to be *Hölder*, and we denote by $\mathcal{H}(\Omega)$ the set of Hölder functions on Ω. Clearly $\mathcal{C}(\Omega) \subset \mathcal{S}(\Omega) \subset \mathcal{H}(\Omega)$.

By D_n we denote the set of all n-tuples $(i_1, i_2, \ldots, i_n)$ (called words of length n) $n \geqslant 1$, which are admissible with respect to Ω, i.e., there exists a sequence $(i'_1, i'_2, \ldots) \in \Omega$ such that $i'_1 = i_1, i'_2 = i_2, \ldots, i'_n = i_n$. The word $\emptyset$ of length zero is called the empty word. Let D_0 be the set consisting of the empty word, and $D = \bigcup_{n\geqslant 0} D_n$. For any $\omega = (\omega_1, \omega_2, \ldots, \omega_{|\omega|}) \in D$, where $|\omega|$ denotes the length of ω, by $[\omega]$ we mean the set of all sequences in Ω with the first $|\omega|$-tuple $(\omega_1, \omega_2, \ldots, \omega_{|\omega|})$, and it is called a *cylinder set* in D of length $|\omega|$. For $\omega \in D \cup \Omega$, if n does not exceed the length of ω by $\omega|_n$ we mean $\omega|_n = (\omega_1, \omega_2, \ldots, \omega_n)$ and $\omega|_0 = \emptyset$, ω is called an extension of $\tau \in D$, written as $\tau \prec \omega$, if $\omega|_{|\tau|} = \tau$. For $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ and $\tau = (\tau_1, \tau_2, \ldots, \tau_p)$ in D, if $M(\omega_n, \tau_1) = 1$ by $\omega\tau$ we mean $\omega\tau = (\omega_1, \omega_2, \ldots, \omega_n, \tau_1, \ldots, \tau_p)$. Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be a set of numbers, called ratio coefficients, such that $0 < \lambda_k < 1$ for $1 \leqslant k \leqslant N$. For $\omega = (\omega_1, \omega_2, \ldots, \omega_{|\omega|}) \in D$ let us write

$$\omega^- = \begin{cases} \emptyset, & |\omega| = 1, \\ (\omega_1, \omega_2, \ldots, \omega_{|\omega|-1}), & |\omega| > 1, \end{cases}$$

$$\lambda_\omega = \begin{cases} 1, & \omega = \emptyset, \\ \lambda_{\omega_1}\lambda_{\omega_2}\cdots\lambda_{\omega_{|\omega|}}, & |\omega| \geqslant 1. \end{cases}$$

The topological entropy $h(\Omega)$ of the shift space Ω is given by $h(\Omega) = \lim_{n\to\infty} \frac{1}{n}\log_2(\#D_n)$, where $\#D_n$ represents the cardinality of D_n (cf. [12]). By our assumption Ω is nonempty, and so $1 \leqslant \#D_n \leqslant N^n$ which implies $0 \leqslant h(\Omega) \leqslant N$. In the paper, we assume that $h(\Omega)$ is positive which we need in proving Lemma 3.2.

Let $J \subset \mathbb{R}^d$ be a compact set such that $J = \mathrm{cl}(\mathrm{int}\,J)$. Then a collection $\mathcal{F} = \{J_\sigma\colon \sigma \in D\}$ of subsets of $\mathbb{R}^d$ is said to fulfill the Moran structure provided it satisfies the following Moran structure conditions (MSC):

$(M1)$ $J_\emptyset = J$.
$(M2)$ For any $\sigma \in D$, J_σ is geometrically similar to J, i.e., there exists a similarity mapping $S_\sigma : \mathbb{R}^d \to \mathbb{R}^d$ such that $J_\sigma = S_\sigma(J)$.
$(M3)$ For any $k \geqslant 1$ and $(i_1, i_2, \ldots, i_k) \in D_k$, the sets $J_{i_1\cdots i_k}$ are subsets of $J_{i_1\cdots i_{k-1}}$, and $\mathrm{int}(J_{i_1\cdots i_k}) \cap \mathrm{int}(J_{i'_1\cdots i'_k}) = \emptyset$ if $(i_1, \ldots, i_k) \neq (i'_1, \ldots, i'_k)$, where $\mathrm{int}(A)$ represents the interior of a set A.
$(M4)$ For any $k \geqslant 1$ and $(i_1, i_2, \ldots, i_k) \in D_k$, $\frac{|J_{i_1\cdots i_k}|}{|J|} = \lambda_{i_1\cdots i_k}$, where $|A|$ denotes the diameter of a set A.

For $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_k) \in D_k$ let us write

$$S_\sigma = \begin{cases} Id_{\mathbb{R}^d}, & \text{if } k = 0, \\ S_{\sigma_1} \circ S_{\sigma_2} \circ \cdots \circ S_{\sigma_k}, & \text{if } k \geqslant 1. \end{cases}$$

Since given $\sigma = (\sigma_i)_{i=1}^\infty \in \Omega$ the diameters of the compact sets $J_{\sigma|_k}, k \geqslant 1$ converge to zero and since they form a descending family, the set

$$\bigcap_{k=0}^{\infty} J_{\sigma|_k}$$

is a singleton and therefore, if we denote its element by $\pi(\sigma)$, this defines the coding map $\pi : \Omega \to J$. The main object of our interest is the limit set

$$E = \pi(\Omega) = \bigcup_{\sigma\in\Omega} \bigcap_{k=1}^{\infty} J_{\sigma|_k}.$$

The set $E := E(\mathcal{F})$ is called the Moran set associated with the collection $\mathcal{F}$. Let $\mathcal{F}_k = \{J_\sigma\colon \sigma \in D_k\}$, and $\mathcal{F} = \bigcup_{k\geqslant 0} \mathcal{F}_k$. The elements of $\mathcal{F}_k$ are called the basic elements of order k, and the elements of $\mathcal{F}$ are called the basic elements of the Moran set E. Note that the set E is a Cantor like set, i.e., it is a perfect, nowhere dense and totally disconnected set. The placements of the basic elements are arbitrary. Moreover, for any $k \geqslant 1$ the set E satisfies the following invariance equality:

$$E = \bigcup_{\sigma\in D_k} S_\sigma(E), \tag{2}$$

where S_σ are the similarity mappings that arise in the Moran structure condition ($M2$). Let us assume that the collection $\mathcal{F} = \{J_\sigma\colon \sigma \in D\}$ satisfies the *open set condition* (OSC): there exists a bounded nonempty open set $U \subset J$ such that

$$\bigcup_{\sigma \in D_k} S_\sigma(U) \subset U \quad \text{and} \quad S_\sigma(U) \cap S_{\sigma'}(U) = \emptyset,$$

for each pair $\sigma, \sigma' \in D_k$ with $\sigma \neq \sigma'$, $k \geqslant 1$. Furthermore, the collection satisfies the *strong open set condition* (SOSC) if U can be chosen such that $U \cap E \neq \emptyset$ where $E := E(\mathcal{F})$ is the Moran set. Given a function $f : \Omega \to \mathbb{R}$, we define the topological pressure of f with respect to the shift map $\sigma : \Omega \to \Omega$ by

$$P_\Omega(f) = \lim_{n\to\infty} \frac{1}{n} \log\biggl(\sum_{\omega \in D_n} \exp\Bigl(\sup_{\tau \in [\omega]} S_n f(\tau)\Bigr)\biggr),$$

where $S_n f(\tau) = \sum_{j=0}^{n-1} f \circ \sigma^j(\tau)$.

If $\phi : \Omega \to \mathbb{R}$ is continuous then a Borel probability measure $\tilde{m}$ is called a Gibbs state for ϕ if there exists a constant $Q \geqslant 1$ such that for every $\omega \in D$ and every $\tau \in [\omega]$

$$Q^{-1} \leqslant \frac{\tilde{m}[\omega]}{\exp(-|\omega| P_\Omega(\phi) + S_{|\omega|}(\phi(\tau)))} \leqslant Q.$$

It is known that for a topologically mixing subshift of finite type Gibbs measure exists for any Hölder continuous function ϕ, and it is unique and coincides with the equilibrium measure μ_ϕ for ϕ, i.e.,

$$P_\Omega(\phi) = \sup_\nu \biggl(h_\nu(\sigma) + \int_\Omega \phi \, d\nu \biggr) = h_{\mu_\phi}(\sigma) + \int_\Omega \phi \, d\mu_\phi,$$

where the supremum is taken over all σ-invariant ergodic Borel probability measures ν with $\nu(\Omega) = 1$, and $h_\nu(\sigma)$ is the entropy of ν with respect to σ (cf. [13]).

Let us now state the following well-known lemma (cf. [14]).

Lemma 2.1. *There exists a positive Hölder continuous function ψ on Ω such that*

$$P_\Omega(\log \psi) = 0.$$

Let ψ be a positive Hölder continuous function as defined in Lemma 2.1. For $u, v \in \mathbb{R}$ let us now define the two parameter family of functions $\phi_{u,v}$ on Ω by $\phi_{u,v}(\omega) = u \log \lambda_{\omega_1} + v \log \psi(\omega) = \log \lambda_{\omega_1}^u \psi(\omega)^v$. Note that for $u, v \in \mathbb{R}$ both the functions $\omega = (\omega_1, \omega_2, \ldots) \mapsto u \log \lambda_{\omega_1}$ and $\omega = (\omega_1, \omega_2, \ldots) \mapsto v \log \psi(\omega)$ are Hölder continuous. Since sum of two Hölder continuous functions is Hölder continuous, the function $\phi_{u,v}$ is Hölder continuous on Ω, and so the Gibbs measure exists for the function $\phi_{u,v}$ for $u, v \in \mathbb{R}$. Let $Q(u, v)$ be the topological pressure of $\phi_{u,v}$ with respect to the shift map σ on Ω, i.e.,

$$Q(u, v) = \lim_{n\to\infty} \frac{1}{n} \log\Biggl(\sum_{\omega \in D_n} \sup_{\tau \in [\omega]} \prod_{k=0}^{n-1} \lambda_{\omega_k}^u \psi\bigl(\sigma^k(\tau)\bigr)^v \Biggr).$$

Let $\hat{\mu} := \hat{\mu}(u, v)$ be the Gibbs measure corresponding to the Hölder continuous function $\phi_{u,v}$ for $u, v \in \mathbb{R}$. Then there exists a constant $K := K(u, v) \geqslant 1$ such that for any $\omega \in D$ and $\tau \in [\omega]$ we have

$$K^{-1} \leqslant \frac{\hat{\mu}[\omega]}{\exp(-|\omega| Q(u, v) + S_{|\omega|}(\phi_{u,v}(\tau)))} \leqslant K,$$

i.e.,

$$K^{-1} \exp\bigl(-|\omega| Q(u, v) + S_{|\omega|}\bigl(\phi_{u,v}(\tau)\bigr)\bigr) \leqslant \hat{\mu}[\omega] \leqslant K \exp\bigl(-|\omega| Q(u, v) + S_{|\omega|}\bigl(\phi_{u,v}(\tau)\bigr)\bigr). \tag{3}$$

Note that for any $\omega\omega' \in D$ with $|\omega| = n$, $|\omega'| = p$ we have

$$S_{n+p}\bigl(\phi_{u,v}(\tau)\bigr) = \sum_{j=0}^{n+p-1} \phi_{u,v}\bigl(\sigma^j(\tau)\bigr) = S_n\bigl(\phi_{u,v}(\tau')\bigr) + S_p\bigl(\phi_{u,v}(\tau'')\bigr),$$

for $\tau \in [\omega\omega']$, $\tau' \in [\omega]$ and $\tau'' \in [\omega']$. Hence (3) implies

$$K^{-1} \leqslant \frac{\hat{\mu}[\omega\omega']}{\exp(-n Q(q, t) + S_n(\phi_{q,t}(\tau'))) \exp(-p Q(q, t) + S_p(\phi_{q,t}(\tau'')))} \leqslant K,$$

i.e.,

$$K^{-3}\hat{\mu}[\omega]\hat{\mu}\big[\omega'\big] \leqslant \hat{\mu}\big[\omega\omega'\big] \leqslant K^{3}\hat{\mu}[\omega]\hat{\mu}\big[\omega'\big]. \tag{4}$$

Let us now define the function $P(q,t) := P(u,v,q,t)$ for $q,t \in \mathbb{R}$ as follows

$$P(q,t) = \lim_{n\to\infty}\frac{1}{n}\log\sum_{\omega\in D_n}\lambda_\omega^t\big(\hat{\mu}[\omega]\big)^q. \tag{5}$$

The limit above exists by the standard theory of subadditive sequences since for any $q,t\in\mathbb{R}$ we have

$$\begin{aligned} K^{-3|q|}\sum_{\omega\in D_n}\lambda_\omega^t\big(\hat{\mu}[\omega]\big)^q\sum_{\omega'\in D_p}\lambda_{\omega'}^t\big(\hat{\mu}\big[\omega'\big]\big)^q &\leqslant \sum_{\omega\omega'\in D_{np}}\lambda_{\omega\omega'}^t\big(\hat{\mu}\big[\omega\omega'\big]\big)^q \\ &\leqslant K^{3|q|}\sum_{\omega\in D_n}\lambda_\omega^t\big(\hat{\mu}[\omega]\big)^q\sum_{\omega'\in D_p}\lambda_{\omega'}^t\big(\hat{\mu}\big[\omega'\big]\big)^q, \end{aligned}$$

where

$$|q| = \begin{cases} q & \text{if}\, q \geqslant 0, \\ -q & \text{if}\, q < 0. \end{cases}$$

The function $P(q,t)$ for $q,t\in\mathbb{R}$ is called the *topological pressure* corresponding to the Gibbs measure and the Moran construction. The following proposition states the well-known properties of the function $P(q,t)$ (cf. [3,17]).

Proposition 2.2.

(i) *$P(q,t):\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous.*
(ii) *$P(q,t)$ is strictly decreasing in each variable separately.*
(iii) *For fixed q we have*

$$\lim_{t\to+\infty}P(q,t) = -\infty \quad \textit{and} \quad \lim_{t\to-\infty}P(q,t) = +\infty.$$

(iv) *$P(q,t)$ is convex: if $q_1,q_2,t_1,t_2\in\mathbb{R}$, $a_1,a_2\geqslant 0$, $a_1+a_2=1$, then*

$$P(a_1q_1+a_2q_2,a_1t_1+a_2t_2)\leqslant a_1P(q_1,t_1)+a_2P(q_2,t_2).$$

Now for fixed q, the function $P(q,t)$ is a continuous function of t. Its value ranges from $-\infty$ (when $t\to+\infty$) to $+\infty$ (when $t\to-\infty$). Therefore, by the intermediate value theorem there is a real number β such that $P(q,\beta)=0$. The solution β is unique, since $P(q,\cdot)$ is strictly decreasing. This defines β implicitly as a function of q: for each q there is a unique $\beta=\beta(q)$ such that $P(q,\beta(q))=0$.

The following proposition gives the well-known properties of the function $\beta(q)$ (cf. [3,17]).

Proposition 2.3. *Let $\beta=\beta(q)$ be defined by $P(q,\beta(q))=0$. Then*

(i) *β is a continuous function of the real variable q.*
(ii) *β is strictly decreasing: if $q_1<q_2$, then $\beta(q_1)>\beta(q_2)$.*
(iii) *$\lim_{q\to-\infty}\beta(q)=+\infty$ and $\lim_{q\to+\infty}\beta(q)=-\infty$.*
(iv) *β is a convex function: if $q_1,q_2,a_1,a_2\in\mathbb{R}$ with $a_1,a_2\geqslant 0$ and $a_1+a_2=1$, then*

$$\beta(a_1q_1+a_2q_2)\leqslant a_1\beta(q_1)+a_2\beta(q_2).$$

The function $\beta(q)$ is sometimes denoted by $T(q)$ and called the *temperature function*. A more general discussion of this function can be found in [9], where our $\beta(q)$ function would correspond to $-\tau(q)$ in their notation.

Remark 2.4. If $q=0$, then $P(q,\beta(q))=0$ implies

$$0=\lim_{n\to\infty}\frac{1}{n}\log\sum_{\omega\in D_n}\lambda_\omega^{\beta(0)}=\lim_{n\to\infty}\frac{1}{n}\log\left(\sum_{i=1}^N\lambda_i^{\beta(0)}\right)^n=\log\sum_{i=1}^N\lambda_i^{\beta(0)},$$

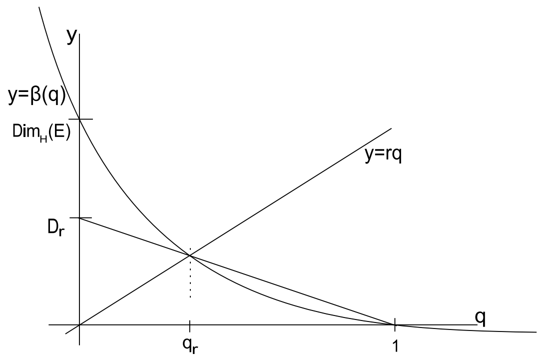

Fig. 1. To determine D_r first find the point of intersection of $y = \beta(q)$ and the line $y = rq$. Then D_r is the y-intercept of the line through this point and the point $(1, 0)$.

and so

$$\sum_{i=1}^{N} \lambda_i^{\beta(0)} = 1.$$

Hence, $\beta(0)$ gives the Hausdorff dimension $\mathrm{Dim}_H(E)$ of the Moran set E (cf. [10]). Note that

$$P(1,0) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega\in D_n} \hat{\mu}[\omega] = \lim_{n\to\infty} \frac{1}{n} \log 1 = 0,$$

and hence $\beta(1) = 0$ (see Fig. 1).

For any $\tau = (\tau_1, \tau_2, \ldots, \tau_k) \in D_k$ by E_τ we mean $E_\tau = S_\tau(E)$, which is called a cylinder set in E of length k. By $\mathcal{D}_k$ we denote the collection of all cylinder sets in E of length k. Let $\mathcal{D} = \bigcup_{k\geqslant 0} \mathcal{D}_k$. Clearly the Borel σ-algebra on E is generated by $\mathcal{D}$. Let $\mu := \mu(u, v) = \hat{\mu} \circ \pi^{-1}$. Then μ is called the image measure of $\hat{\mu}$ under the coding map π on the Moran set E such that for any Borel $B \subset E$

$$\mu(B) = \inf\left\{ \sum_i \mu(U_i):\ B \subset \bigcup_i U_i,\ U_i \in \mathcal{D} \right\}.$$

For this measure μ we will determine the quantization dimension function and its relationship with the temperature function $\beta(q)$ of the thermodynamic formalism that arises in multifractal analysis.

3. Main result

The relationship between the quantization dimension function D_r and the temperature function $\beta(q)$ for the probability measure μ, where the temperature function is the Legendre transform of the $f(\alpha)$ curve (the definitions of $f(\alpha)$ and Legendre transform are given in [2]) is given by the following theorem. For a graphical description see Fig. 1.

Theorem 3.1. *Let μ be the image measure on the Moran set E of the Gibbs measure $\hat{\mu}$ corresponding to the Hölder continuous function $\phi_{u,v}$ on the coding space under the coding map. Let $\beta = \beta(q)$ be the temperature function of the thermodynamic formalism. For each $r \in (0, +\infty)$ choose q_r such that $\beta(q_r) = rq_r$. Then the quantization dimension function for the probability measure μ is given by*

$$D_r = \frac{\beta(q_r)}{1 - q_r}.$$

Lemma 3.2. *Let $0 < r < +\infty$ be fixed. Then there exists exactly one number $\kappa_r \in (0, +\infty)$ such that*

$$\lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega\in D_n} \left(\lambda_\omega^r \hat{\mu}[\omega]\right)^{\frac{\kappa_r}{r+\kappa_r}} = 0.$$

Proof. From Eq. (5) we have

$$P(t, rt) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega\in D_n} \left(\lambda_\omega^r \hat{\mu}[\omega]\right)^t.$$

Proposition 2.2 says that $P(t,rt)$ is continuous and strictly decreasing, and hence there exists a unique $t \in \mathbb{R}$ such that $P(t,rt)=0$. If $t=0$ then, $P(0,0)=\lim_{n\to\infty}\frac{1}{n}\log\sum_{\omega\in D_n}1=\lim_{n\to\infty}\frac{1}{n}\log(\#D_n)=(\log 2)\lim_{n\to\infty}\frac{1}{n}\log_2(\#D_n)=(\log 2)h(\Omega)>0$; and if $t=1$ then, $P(1,r1)=\lim_{n\to\infty}\frac{1}{n}\log\sum_{\omega\in D_n}\lambda_\omega^r\hat{\mu}[\omega]\leqslant\log\lambda_{\max}^r+\lim_{n\to\infty}\frac{1}{n}\log\sum_{\omega\in D_n}\hat{\mu}[\omega]=r\log\lambda_{\max}<0$, where $\lambda_{\max}=\max\{\lambda_j\colon 1\leqslant j\leqslant N\}$. Therefore, the unique $t\in\mathbb{R}$ for which $P(t,rt)=0$ must lie between 0 and 1. Then $\kappa_r=\frac{rt}{1-t}$ satisfies the conclusion of the lemma. □

Lemma 3.3. *Let $0<r<+\infty$ and κ_r be as in Lemma 3.2. Then for any $n\geqslant 1$ we have*

$$K^{-\frac{2\kappa_r}{r+\kappa_r}}\leqslant\sum_{\omega\in D_n}\left(\lambda_\omega^r\hat{\mu}[\omega]\right)^{\frac{\kappa_r}{r+\kappa_r}}\leqslant K^{\frac{2\kappa_r}{r+\kappa_r}}.$$

Proof. For any $\omega\in D$, let $s_\omega=\lambda_\omega^r\hat{\mu}[\omega]$. Then by (4) for $\omega\in D_n,\tau\in D_p$ $(n,p\geqslant 1)$ with $\omega\tau\in D$, we have $K^{-3}s_\omega s_\tau\leqslant s_{\omega\tau}\leqslant K^3 s_\omega s_\tau$. Since $K\geqslant 1$, it is also true that $K^{-4}s_\omega s_\tau\leqslant s_{\omega\tau}\leqslant K^4 s_\omega s_\tau$. By the standard theory of subadditive sequences, $\lim_{n\to\infty}\frac{1}{n}\log\sum_{\omega\in D_n}s_\omega^t$ exists for any $t\in\mathbb{R}$. Let us denote this limit by $h(t)$. Hence for $t\geqslant 0$ we have

$$h(t)=\lim_{p\to\infty}\frac{1}{np}\log\sum_{\omega\in D_{np}}s_\omega^t,$$

and so

$$\lim_{p\to\infty}\frac{1}{np}\log\left(\sum_{\omega\in D_n}s_\omega^t K^{-2t}\right)^p\leqslant h(t)\leqslant\lim_{p\to\infty}\frac{1}{np}\log\left(\sum_{\omega\in D_n}s_\omega^t K^{2t}\right)^p,$$

which implies

$$\frac{1}{n}\log\sum_{\omega\in D_n}s_\omega^t K^{-2t}\leqslant h(t)\leqslant\frac{1}{n}\log\sum_{\omega\in D_n}s_\omega^t K^{2t}$$

and therefore,

$$e^{nh(t)}K^{-2t}\leqslant\sum_{\omega\in D_n}s_\omega^t\leqslant e^{nh(t)}K^{2t}.$$

Now substitute $t=\frac{\kappa_r}{r+\kappa_r}$ and note that $h(t)=0$ to obtain the assertion. □

We call $\Gamma\subset D$ a finite maximal antichain if Γ is a finite set of words in D, such that every sequence in Ω is an extension of some word in Γ, but no word of Γ is an extension of another word in Γ. Of course, this requires that the index set $\{1,2,\dots,N\}$ is finite. We will make this assumption in the remainder of this paper. By $|\Gamma|$ we denote the cardinality of Γ. Note that from the definition of Γ it follows that finite maximal antichain does not contain the empty word $\emptyset$ as all words are extension of $\emptyset$.

Lemma 3.4. *Let Γ be a finite maximal antichain. Then,*

(a) $$K^{-3}\sum_{\omega\in\Gamma}\hat{\mu}[\omega]\mu\circ S_\omega^{-1}\leqslant\mu\leqslant K^3\sum_{\omega\in\Gamma}\hat{\mu}[\omega]\mu\circ S_\omega^{-1},$$

and

(b) $$\sum_{\omega\in\Gamma}\left(\lambda_\omega^r\hat{\mu}[\omega]\right)^{\frac{\kappa_r}{r+\kappa_r}}\leqslant K^{\frac{5\kappa_r}{r+\kappa_r}},$$

where κ_r is as in Lemma 3.2.

Proof. (a) Let $n\in\mathbb{N}$ and $n\geqslant\max\{|\omega|\colon\omega\in\Gamma\}$. Since E satisfies the invariance equality (2), it is enough to prove that for any $E_\tau\in\mathcal{D}_k$ with $k\geqslant n$,

$$\mu(E_\tau)\leqslant K^3\sum_{\omega\in\Gamma}\hat{\mu}[\omega]\mu\circ S_\omega^{-1}(E_\tau).$$

Since Γ is a finite maximal antichain, for $\tau\in D_k$ $(k\geqslant n)$ there exists $x\in\Gamma$ such that $\tau=xy$ for some $y\in D$. Then, $E_\tau=E_{xy}=S_{xy}(E)=S_x(S_y(E))=S_x(E_y)$. Hence,

$$\sum_{\omega\in\Gamma}\hat{\mu}[\omega]\mu\circ S_\omega^{-1}(E_\tau)=\sum_{\omega\in\Gamma}\hat{\mu}[\omega]\mu\circ S_\omega^{-1}\big(S_x(E_y)\big)=\hat{\mu}[x]\mu\circ S_x^{-1}\big(S_x(E_y)\big)=\hat{\mu}[x]\mu(E_y)$$
$$=\hat{\mu}[x]\big(\hat{\mu}\circ\pi^{-1}\big)(E_y)=\hat{\mu}[x]\hat{\mu}[y]\geqslant K^{-3}\hat{\mu}[xy]=K^{-3}\hat{\mu}[\tau]=K^{-3}\mu(E_\tau),$$

which implies $\mu(E_\tau)\leqslant K^3\sum_{\omega\in\Gamma}\hat{\mu}[\omega]\mu\circ S_\omega^{-1}(E_\tau)$ for any $E_\tau\in\mathcal{D}_k$ with $k\geqslant n$. Similarly, it can be proved that $K^{-3}\sum_{\omega\in\Gamma}\hat{\mu}[\omega]\mu\circ S_\omega^{-1}(E_\tau)\leqslant\mu(E_\tau)$ for any $E_\tau\in\mathcal{D}_k$ with $k\geqslant n$, completing the proof of (a).

To prove (b) let us proceed as follows: let $m=\min\{|\omega|\colon\ \omega\in\Gamma\}$. Since Γ does not contain the empty word, we have $m\geqslant 1$. Then for each $\omega\in\Gamma$ there exists $\tau(\omega)\in D$ with $|\tau(\omega)|=m$ and $\tau(\omega)\prec\omega$, i.e., there exists $x(\omega)\in D$ such that $\omega=\tau(\omega)x(\omega)$. Now for any $\omega\in\Gamma$ we can write

$$\lambda_\omega=\lambda_{\tau(\omega)}\lambda_{x(\omega)}\leqslant\lambda_{\tau(\omega)}\quad\text{and}\quad\hat{\mu}[\omega]\leqslant K^3\hat{\mu}\big[\tau(\omega)\big]\hat{\mu}\big[x(\omega)\big]\leqslant K^3\hat{\mu}\big[\tau(\omega)\big].$$

Using the above inequalities and Lemma 3.3, we have

$$\sum_{\omega\in\Gamma}\big(\lambda_\omega^r\hat{\mu}[\omega]\big)^{\frac{\kappa_r}{r+\kappa_r}}\leqslant K^{\frac{3\kappa_r}{r+\kappa_r}}\sum_{\omega\in\Gamma}\big(\lambda_{\tau(\omega)}^r\hat{\mu}\big[\tau(\omega)\big]\big)^{\frac{\kappa_r}{r+\kappa_r}}\leqslant K^{\frac{3\kappa_r}{r+\kappa_r}}\sum_{\tau\in D_m}\big(\lambda_\tau^r\hat{\mu}[\tau]\big)^{\frac{\kappa_r}{r+\kappa_r}}\leqslant K^{\frac{5\kappa_r}{r+\kappa_r}}.\qquad\square$$

Lemma 3.5. *Let $\Gamma\subset D$ be a finite maximal antichain, $n\in\mathbb{N}$ with $n\geqslant|\Gamma|$, and $0<r<+\infty$. Then*

$$V_{n,r}(\mu)\leqslant\inf\Big\{K^3\sum_{\omega\in\Gamma}\lambda_\omega^r\hat{\mu}[\omega]V_{n_\omega,r}(\mu)\colon\ 1\leqslant n_\omega,\ \sum_{\omega\in\Gamma}n_\omega\leqslant n\Big\}.$$

Proof. Suppose $n_\omega\geqslant 1$ for each $\omega\in\Gamma$, and $\sum_{\omega\in\Gamma}n_\omega\leqslant n$. For each $\omega\in\Gamma$ let α_ω be an n_ω-optimal set for $V_{n_\omega,r}(\mu)$. Since $|\bigcup_{\omega\in\Gamma}S_\omega(\alpha_\omega)|\leqslant n$, $\mu\leqslant K^3\sum_{\omega\in\Gamma}\hat{\mu}[\omega]\mu\circ S_\omega^{-1}$, we have

$$\begin{aligned}V_{n,r}(\mu)&\leqslant\int d\Big(x,\bigcup S_\omega(\alpha_\omega)\Big)^r d\mu(x)\\&\leqslant K^3\sum_{\omega\in\Gamma}\hat{\mu}[\omega]\int d\Big(x,\bigcup S_\omega(\alpha_\omega)\Big)^r d\big(\mu\circ S_\omega^{-1}\big)(x)\\&\leqslant K^3\sum_{\omega\in\Gamma}\hat{\mu}[\omega]\int d\big(S_\omega(x),S_\omega(\alpha_\omega)\big)^r d\mu(x)\\&=K^3\sum_{\omega\in\Gamma}\lambda_\omega^r\hat{\mu}[\omega]\int d(x,\alpha_\omega)^r d\mu(x)\\&=K^3\sum_{\omega\in\Gamma}\lambda_\omega^r\hat{\mu}[\omega]V_{n_\omega,r}(\mu),\end{aligned}$$

which yields the lemma. $\square$

Proposition 3.6. *Let $0<r<+\infty$ and κ_r be as in Lemma 3.2. Then $\limsup_{n\to\infty}ne_{n,r}^{\kappa_r}<+\infty$.*

Proof. Let $\epsilon_0=\min\{(\lambda_j^r\hat{\mu}[j])^{\frac{\kappa_r}{r+\kappa_r}}\colon\ 1\leqslant j\leqslant N\}$. Then $0<\epsilon_0<1$. Fix $m\in\mathbb{N}$. Choose any $n\in\mathbb{N}$ so that $\frac{m}{n}K^{\frac{3\kappa_r}{r+\kappa_r}}<\epsilon_0^2$, and set $\epsilon=\epsilon_0^{-1}\frac{m}{n}K^{\frac{3\kappa_r}{r+\kappa_r}}$. Then $0<\epsilon<1$. Let $\Gamma=\Gamma(\epsilon)=\{\omega\in D\colon\ (\lambda_\omega^r\hat{\mu}[\omega])^{\frac{\kappa_r}{r+\kappa_r}}<\epsilon\leqslant(\lambda_{\omega^-}^r\hat{\mu}[\omega^-])^{\frac{\kappa_r}{r+\kappa_r}}\}$. Let $\ell=\min\{|\omega|\colon\ \omega\in\Gamma\}$. Then $\ell\geqslant 1$, and using Lemma 3.3 we have

$$\sum_{\omega\in\Gamma}\big(\lambda_\omega^r\hat{\mu}[\omega]\big)^{\frac{\kappa_r}{r+\kappa_r}}\leqslant\sum_{\tau\in D_\ell}\big(\lambda_\tau^r\hat{\mu}[\tau]\big)^{\frac{\kappa_r}{r+\kappa_r}}\leqslant K^{\frac{2\kappa_r}{r+\kappa_r}}$$

and so,

$$K^{\frac{2\kappa_r}{r+\kappa_r}}\geqslant\sum_{\omega\in\Gamma}\big(\lambda_\omega^r\hat{\mu}[\omega]\big)^{\frac{\kappa_r}{r+\kappa_r}}\geqslant\sum_{\omega\in\Gamma}K^{-\frac{\kappa_r}{r+\kappa_r}}\big(\lambda_{\omega^-}^r\hat{\mu}\big[\omega^-\big]\big)^{\frac{\kappa_r}{r+\kappa_r}}\big(\lambda_{\omega_{|\omega|}}^r\hat{\mu}[\omega_{|\omega|}]\big)^{\frac{\kappa_r}{r+\kappa_r}}\geqslant K^{-\frac{\kappa_r}{r+\kappa_r}}\epsilon\epsilon_0|\Gamma|.$$

Hence,

$$|\Gamma|\leqslant K^{\frac{3\kappa_r}{r+\kappa_r}}(\epsilon\epsilon_0)^{-1}=\frac{n}{m}<+\infty,\quad\text{i.e.,}\quad\Gamma\text{ is a finite maximal antichain, and }n\geqslant m|\Gamma|.$$

Hence by the previous lemma, we have

$$\begin{aligned}
V_{n,r}(\mu) &\leqslant K^3 \sum_{\omega\in\Gamma} \lambda_\omega^r \hat{\mu}[\omega] V_{m,r}(\mu) \\
&= K^3 \sum_{\omega\in\Gamma} \big(\lambda_\omega^r \hat{\mu}[\omega]\big)^{\frac{\kappa_r}{r+\kappa_r}} \big(\lambda_\omega^r \hat{\mu}[\omega]\big)^{\frac{r}{r+\kappa_r}} V_{m,r}(\mu) \\
&< K^3 \sum_{\omega\in\Gamma} \big(\lambda_\omega^r \hat{\mu}[\omega]\big)^{\frac{\kappa_r}{r+\kappa_r}} \epsilon^{\frac{r}{\kappa_r}} V_{m,r}(\mu) \\
&\leqslant K^3 K^{\frac{5\kappa_r}{r+\kappa_r}} \epsilon^{\frac{r}{\kappa_r}} V_{m,r}(\mu) \quad \text{(by Lemma 3.4)} \\
&= K^3 K^{\frac{8\kappa_r}{r+\kappa_r}} \epsilon_0^{-\frac{r}{\kappa_r}} \left(\frac{m}{n}\right)^{\frac{r}{\kappa_r}} V_{m,r}(\mu),
\end{aligned}$$

and therefore, $nV_{n,r}^{\frac{\kappa_r}{r}}(\mu) \leqslant K^{3\kappa_r/r} K^{\frac{8\kappa_r^2}{r(r+\kappa_r)}} \epsilon_0^{-1} m V_{m,r}^{\frac{\kappa_r}{r}}(\mu)$. Since for fixed m, this inequality holds for all but a finite number of n, we have

$$\limsup_{n\to\infty} n e_{n,r}^{\kappa_r} \leqslant K^{3\kappa_r/r} K^{\frac{8\kappa_r^2}{r(r+\kappa_r)}} \epsilon_0^{-1} m e_{m,r}^{\kappa_r} < +\infty,$$

and thus the proposition is proved. □

Lemma 3.7. *Let $\Gamma \subset D$ be a finite maximal antichain. Then there exists $n_0 = n_0(\Gamma)$ such that for every $n \geqslant n_0$ there exists a set of positive integers $\{n_\omega := n_\omega(n)\}_{\omega\in\Gamma}$ such that $\sum_{\omega\in\Gamma} n_\omega \leqslant n$ and*

$$u_{n,r} \geqslant K^{-3} \sum_{\omega\in\Gamma} \lambda_\omega^r \hat{\mu}[\omega] u_{n_\omega,r}.$$

Proof. Let U be the open set from the strong open set condition. Then there exists $\tau \in D$ such that $S_\tau(J) \subset U$. Note that here the similarity mappings S_σ for $\sigma \in D$ are the similarity mappings as defined in $(M2)$ of the Moran set construction. Let $\epsilon = d(S_\tau(J), U^c)$ and $\underline{\lambda} = \min_{\omega\in\Gamma}\{\lambda_\omega\}$. Then for $\omega \in \Gamma$ we have $d(S_\omega S_\tau(J), S_\omega(U^c)) = \lambda_\omega d(S_\tau(J), U^c) \geqslant \underline{\lambda}\epsilon$, which implies $d(x, U^c) \geqslant d(x, S_\omega(U^c)) \geqslant \underline{\lambda}\epsilon$ for any $x \in S_\omega(S_\tau(J))$. For each n, let α_n be an n-optimal set for $u_{n,r}$ and let $\delta_n = \max\{d(x, \alpha_n \cup U^c)\colon x \in E\}$. Since $\delta_n \to 0$ as $n \to \infty$ we can choose n_0 such that $\delta_n < \underline{\lambda}\epsilon$ for all $n \geqslant n_0$. Suppose $n \geqslant n_0$ and $x \in S_\omega(S_\tau(E))$. Note that $S_\omega(S_\tau(E)) \subset S_\tau(E) \subset S_\tau(J) \subset U$. Then there exists $a \in \alpha_n \cup U^c$ such that $d(x, \alpha_n \cup U^c) = d(x, a) \leqslant \delta_n < \underline{\lambda}\epsilon$ and so $a \notin S_\omega(U^c)$, i.e., $a \in S_\omega(U)$. Therefore, letting $\alpha_{n_\omega} = \alpha_n \cap S_\omega(U)$, we have $n_\omega := |\alpha_{n_\omega}| \geqslant 1$ and $\sum_{\omega\in\Gamma} n_\omega \leqslant n$. For any $x \in E$, we claim that there exists $y \in \alpha_{n_\omega} \cup S_\omega(U^c)$ such that $d(S_\omega(x), \alpha_{n_\omega} \cup S_\omega(U^c)) = d(x, y)$. If not let $y \notin \alpha_{n_\omega} \cup S_\omega(U^c)$. Then $y \notin \alpha_n \cap S_\omega(U)$ and $y \notin S_\omega(U^c)$, i.e., $z := S_\omega^{-1}(y) \in J$ be such that $z \notin U$ and $z \notin U^c$, which gives a contradiction and thus the claim is true, and it implies $d(S_\omega(x), \alpha_n \cup S_\omega(U^c)) = d(S_\omega(x), \alpha_{n_\omega} \cup S_\omega(U^c))$. Again for any $x \in E$ as $U^c \subset (S_\omega(U))^c = S_\omega(U^c)$, we have $d(S_\omega(x), \alpha_n \cup U^c) \geqslant d(S_\omega(x), \alpha_n \cup S_\omega(U^c))$. Hence,

$$\begin{aligned}
u_{n,r} &= \int d\big(x, \alpha_n \cup U^c\big)^r \, d\mu(x) \\
&\geqslant K^{-3} \sum_{\omega\in\Gamma} \hat{\mu}[\omega] \int d\big(S_\omega(x), \alpha_n \cup U^c\big)^r \, d\mu(x) \\
&\geqslant K^{-3} \sum_{\omega\in\Gamma} \hat{\mu}[\omega] \int d\big(S_\omega(x), \alpha_n \cup S_\omega\big(U^c\big)\big)^r \, d\mu(x) \\
&= K^{-3} \sum_{\omega\in\Gamma} \hat{\mu}[\omega] \int d\big(S_\omega(x), \alpha_{n_\omega} \cup S_\omega\big(U^c\big)\big)^r \, d\mu(x) \\
&= K^{-3} \sum_{\omega\in\Gamma} \lambda_\omega^r \hat{\mu}[\omega] \int d\big(x, S_\omega^{-1}(\alpha_{n_\omega}) \cup U^c\big)^r \, d\mu(x) \\
&\geqslant K^{-3} \sum_{\omega\in\Gamma} \lambda_\omega^r \hat{\mu}[\omega] u_{n_\omega,r}. \qquad \square
\end{aligned}$$

Proposition 3.8. *Let the Moran set construction satisfy the strong open set condition and let $0 < r < +\infty$. Moreover, let κ_r be as in Lemma 3.2. Let $0 < \ell < \kappa_r$. Then $\liminf_{n\to\infty} n e_{n,r}^{\ell} > 0$.*

Proof. Since $0 < \ell < \kappa_r$ and κ_r is unique for which $\lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega\in D_n} (\lambda_\omega^r \hat{\mu}[\omega])^{\frac{\kappa_r}{r+\kappa_r}} = 0$, we have

$$\sum_{\omega\in D_m} \left(\lambda_\omega^r \hat{\mu}[\omega]\right)^{\frac{\ell}{r+\ell}} \to \infty \quad \text{as } m \to \infty.$$

Choose m so that the above sum is greater than 1 and let $\Gamma = \{\omega \in D\colon\ |\omega| = m\}$. Then Γ is a finite maximal antichain. By the previous lemma we have n_0, and for $n \geqslant n_0$ the numbers $\{n_\omega := n_\omega(n)\}_{\omega\in\Gamma}$ which satisfy the conclusion of that lemma. Set $c = \min\{n^{r/\ell} u_{n,r}\colon\ n \leqslant n_0\}$. Clearly each $u_{n,r} > 0$ and hence $c > 0$. Suppose $n \geqslant n_0$ and $k^{r/\ell} u_{k,r} \geqslant c$ for all $k < n$. Using the previous lemma, we have

$$\begin{aligned} n^{r/\ell} u_{n,r} &\geqslant n^{r/\ell} K^{-3} \sum_{\omega\in\Gamma} \lambda_\omega^r \hat{\mu}[\omega] u_{n_\omega,r} \\ &= n^{r/\ell} K^{-3} \sum_{\omega\in\Gamma} \lambda_\omega^r \hat{\mu}[\omega] \big(n_\omega(n)\big)^{-r/\ell} \big(n_\omega(n)\big)^{r/\ell} u_{n_\omega,r} \\ &\geqslant cK^{-3} \sum_{\omega\in\Gamma} \lambda_\omega^r \hat{\mu}[\omega] \left(\frac{n_\omega(n)}{n}\right)^{-r/\ell}. \end{aligned}$$

Using Hölder's inequality (with exponents less than 1) we have

$$n^{r/\ell} u_{n,r} \geqslant cK^{-3} \left(\sum_{\omega\in\Gamma} \left(\lambda_\omega^r \hat{\mu}[\omega]\right)^{\ell/(r+\ell)} \right)^{(1+r/\ell)} \left(\sum_{\omega\in\Gamma} \left(\frac{n_\omega(n)}{n}\right)^{(-r/\ell)(-\ell/r)} \right)^{-r/\ell}.$$

By our choice of Γ, which depends only on ℓ and not on n, and the fact that $\sum_{\omega\in\Gamma} n_\omega(n) \leqslant n$, we see that $n^{r/\ell} u_{n,r} \geqslant cK^{-3}$. Hence by induction, we have

$$\liminf_{n\to\infty} n u_{n,r}^{\ell/r} \geqslant \left(cK^{-3}\right)^{\ell/r} > 0, \quad \text{i.e.,} \quad \liminf_{n\to\infty} n e_{n,r}^{\ell} > 0. \qquad \square$$

Proof of Theorem 3.1. From Proposition 11.3 of [5] we know:

(a) If $0 \leqslant t < \underline{D}_r < s$ then

$$\lim_{n\to\infty} n e_{n,r}^t = +\infty \quad \text{and} \quad \liminf_{n\to\infty} n e_{n,r}^s = 0.$$

(b) If $0 \leqslant t < \overline{D}_r < s$ then

$$\limsup_{n\to\infty} n e_{n,r}^t = +\infty \quad \text{and} \quad \lim_{n\to\infty} n e_{n,r}^s = 0.$$

From (a) and Proposition 3.8 we have, $\ell \leqslant \underline{D}_r$ whenever $\ell < \kappa_r$. Hence $\kappa_r \leqslant \underline{D}_r$. From (b) and Proposition 3.6 we have, $\overline{D}_r \leqslant \kappa_r$. Hence $\kappa_r \leqslant \underline{D}_r \leqslant \overline{D}_r \leqslant \kappa_r$, i.e., the quantization dimension D_r exists and $D_r = \kappa_r$. Note that for $q_r = \frac{\kappa_r}{r+\kappa_r}$ and $\beta(q_r) = rq_r$ we have $D_r = \frac{\beta(q_r)}{1-q_r}$. This completes the proof of the theorem. $\square$

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